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Differential Constants of Motion for Systems of Free Gravitating Particles. I. Newton's Theory

M. ENOSH[1] and A. KOVETZ

Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel

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Abstract

Differential constants of motion for systems of free gravitating particles in the Newtonian frame are first defined and then determined. It is shown that they are all implied by the existence of the first integral invariants of Poincaré known from classical mechanics, or by the circulation theorem known from hydrodynamics. It is proved further that the restriction to vacuum conditions does not change the set of differential constants of motion. Another consequence is that nothing can be inferred from local (in space and time) measurements about the displacement, velocity, and orientation of a laboratory in free fall relative to a fixed Galilean frame.

1. *Introduction*

We intend to study one aspect of the motion of a continuum consisting of freely gravitating, noncolliding particles moving in accordance with Newton's theory. Along the world-line of any particle surrounded by others, all in free fall, it is possible to speak of certain quantities, the differential quantities. At every event on such a world-line these are functions defined on certain domains of the following arguments: the (proper) time (of the chosen particle), the functions that describe space-time and the particles' motion and their derivatives up to a certain order; all these are calculated at the given event. We try to adopt here as general an approach as possible. However, in order to derive meaningful results we should consider only those quantities that are determined by the physical and mathematical structure of the system, namely, the covariant quantities. Then, our main purpose is to select out of these the differential constants of motion (DCMs). The DCMs are characterized by the further res-

[1] Present address: Department of Theoretical Physics, The Hebrew University, Jerusalem, Israel.

triction of being constant along the histories of all the particles in every continuum in every possible (Newtonian) gravitational field.

Throughout this paper, lower case Latin and Greek and capital Latin indices take the ranges $\{0, 1, 2, 3\}$, $\{1, 2, 3\}$, and $\{\bar{1}, \bar{2}, \ldots, \bar{6}\}$, respectively, except when stated otherwise. Partial derivatives are sometimes denoted by a diagonal stroke (e.g., $\partial\psi/\partial x^i = \psi/i$, $\partial r_\alpha/\partial d^A = r_{\alpha/A}$). Parentheses and square brackets around indices denote the symmetric and the antisymmetric part, respectively. 3-vectors are denoted by boldface letters (e.g., $\mathbf{v}$), and scalar products of these by $(\mathbf{a} \cdot \mathbf{b})$. The general summation convention is strictly kept (a letter appearing twice, no matter where, as an index of a product should be automatically summed).

In order to find the DCMs it turns out that one has to solve a system of homogeneous linear partial differential equations of the first order for a single unknown function. This theory is based on the famous Frobenius' integration theorem (Flanders, 1963), and the technique of treatment is outlined, for example, in Schouten (1954). We shall make use of this technique here. In order to describe our operations we found it useful and economical to introduce the following convention.

Let $F(y)$ satisfy the equations

$$\text{(a)} \qquad a^i \frac{\partial}{\partial y^i} F = 0$$

$$\text{(b)} \qquad b^i \frac{\partial}{\partial y^i} F = 0$$

(Here and in the following the indices $i, j, \ldots$ run over any finite set.) Then F satisfies the following equation (c), obtained from (a) and (b) by means of a process which we call "crossing of (a) and (b)":

$$\text{(c)} \qquad a^j \frac{\partial}{\partial y^j}\left(b^i \frac{\partial}{\partial y^i} F\right) - b^j \frac{\partial}{\partial y^j}\left(a^i \frac{\partial}{\partial y^i} F\right) = 0$$

(c) is again a homogeneous linear differential equation of the first order:

$$\text{(c)} \qquad c^i \frac{\partial}{\partial y^i} F = 0, \qquad c^i \equiv a^j b^i/j - b^j a^i/j$$

We shall write symbolically $[\mathrm{a}, \mathrm{b}] = (\mathrm{c})$.

2. *Differential Quantities and the Definition of a Differential Constant of Motion (DCM)*

Let x_α be arbitrary Galilean coordinates of space and $t = x_0$ the time. Six parameters, $(d) \equiv (d^A)$, serve to identify all the possible motions, $\mathbf{r}(t; d)$, of free particles. According to Newton's theory of gravitation

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} = \operatorname{grad} \phi \tag{2.1}$$

where $\phi = \phi(t, \mathbf{r})$ is a scalar function (with respect to Galilean transformations), which is determined in a given physical system up to gauge transformations $\phi \to \phi + f(t)$, where f depends only on the time. At first we assume that ϕ is arbitrary. The physical inequality $\Delta\phi \leqslant 0$ (following from $\Delta\phi = -4\pi\rho$), may be assumed. The discussion and the results are insensitive to this assumption. Later in Section 4 we assume the vacuum condition $\Delta\phi = 0$.

Differential quantities (along the history of a certain chosen particle d) are constructed out of t and the derivatives of ϕ and $\mathbf{r}$. Let us define

$$v_\alpha \equiv \frac{\partial r_\alpha}{\partial t}, \qquad r_{\alpha A_1 \cdots A_n} \equiv r_{\alpha/A_1/\cdots/A_n},$$

$$v_{\alpha A_1 \cdots A_n} \equiv v_{\alpha/A_1/\cdots/A_n}, \qquad \phi_{i_1 \cdots i_n} \equiv \phi_{/i_1/i_2/\cdots/i_n} \tag{2.2}$$

It is easy to show that apart from the symmetries

$$\mathbf{r}_{A_1 \cdots A_n} = \mathbf{r}_{(A_1 \cdots A_n)}, \qquad \mathbf{v}_{A_1 \cdots A_n} = \mathbf{v}_{(A_1 \cdots A_n)},$$

$$\phi_{i_1 \cdots i_n} = \phi_{(i_1 \cdots i_n)} \tag{2.3}$$

the quantities

$$\{t, r_\alpha, v_\alpha, r_{\alpha A}, v_{\alpha A}, r_{\alpha AB}, v_{\alpha AB}, \ldots, \phi, \phi_i, \phi_{ij}, \ldots\} \tag{2.4}$$

are functionally independent; moreover, taking (2.1) into consideration, it follows that every derivative of ϕ and $\mathbf{r}$ is a function of the quantities (2.4). Therefore, *the quantities* (2.4) *serve as a basis* (*in the functional sense*) *for the* (*not necessarily covariant*) *differential quantities.* We shall refer to them later as *the basic differential quantities*, or just *the basic quantities.*

In addition to (2.3) there is still another restriction on the basic differential quantities which expresses the claim that $\mathbf{r}(t, d)$ includes (at least locally) all the possible motions of gravitating particles. But this restriction is an inequality; therefore it does not further reduce the set of functionally independent basic differential quantities. We may write it in the following form:

$$\det(r_{\alpha A} v_{\alpha A}) \equiv \det \begin{pmatrix} (r_{\bar{1}}) & (v_{\bar{1}}) \\ (r_{\bar{2}}) & (v_{\bar{2}}) \\ \vdots & \vdots \\ (r_{\bar{6}}) & (v_{\bar{6}}) \end{pmatrix} \neq 0 \tag{2.5}$$

where (r_A) and (v_A) stand for the rows with the components $r_{\alpha A}$ and $v_{\alpha A}$, respectively.

The above-mentioned independence of the basic differential quantities leads to the following simple classification: A basic quantity of order n possesses exactly n indices of type $A, B, \ldots$, etc. (e.g., $t, r_\alpha, v_\alpha, \phi_{i_1 \cdots i_k}$ are of order zero). We generalize this to other differential quantities: A differential quantity is of order n if as a function of the basic quantities, n is the highest order of its nontrivial arguments.

In accordance with the introduction *we define the DCMs to be those differential quantities that are constant along the histories of all freely falling particles in every parametrized continuum in every possible gravitational field and every Galilean reference frame and gauge of potential* ϕ.

The basic quantities (2.4) are not necessarily covariant: They may depend on the gauge of ϕ and on the Galilean reference frame. However, we dismiss this fact now, hoping we shall be able to characterize the desired covariant quantities in the larger set later on. On the other hand this approach may be advantageous. After all a choice of a certain Galilean frame enriches the mathematical-physical structure of the system. For example, the occurrence, maybe, of any non-Galilean invariant DCM means that it is possible to learn something about the displacement, velocity, and orientation of a laboratory associated with a particle from local measurements, only (!).

In order to find the DCMs we should know the time derivatives of the basic quantities along the particles' histories. We denote this kind of derivative by a dot. Making use of (2.1) and (2.2) we obtain.

$$\dot{t} = 1 \tag{2.6a}$$

$$\dot{r}_{\alpha A_1 \cdots A_n} = v_{\alpha A_1 \cdots A_n} \tag{2.6b}$$

$$\dot{v}_\alpha = \phi_\alpha \tag{2.6c}$$

$$\dot{v}_{\alpha A} = \phi_{\alpha\beta} r_{\beta A} \tag{2.6d}$$

$$\dot{v}_{\alpha AB} = \phi_{\alpha\beta\gamma} r_{\gamma A} r_{\beta B} + \phi_{\alpha\beta} r_{\beta AB} \tag{2.6e}$$

$$\dot{v}_{\alpha A_1 \cdots A_n} = \phi_{\alpha\beta_1 \cdots \beta_n} r_{\beta_1 A_1} \cdots r_{\beta_n A_n} + \cdots + \phi_{\alpha\beta} r_{\beta A_1 \cdots A_n}$$
$$(n = 3, 4, 5, \ldots) \tag{2.6f}$$

$$\dot{\phi}_{i_1 \cdots i_n} = \phi_{i_1 \cdots i_n 0} + \phi_{i_1 \cdots i_n \alpha} v_\alpha \qquad (n = 0, 1, 2, \ldots) \tag{2.6g}$$

where the terms which are not written explicitly in (2.6f), are monomials in the arguments $\{\phi_{\alpha\beta_1 \cdots \beta_k}, r_{\beta A_1 \cdots A_l}\}_{k=2,\ldots,n-1;\, l=1,\ldots,n-1}$ which are linear homogeneous (degree 1 exactly) in the $\{\phi_{\alpha\beta_1 \cdots \beta_k}\}_{k=2,\ldots,n-1}$. In particular, the right-hand side of (2.6f) is independent of the $\{\phi_{0 i_1 \cdots i_k}\}_{k=0,1,2,3,\ldots}$.

Obviously a function F with arguments from (2.4), (a differential quantity), is a DCM if and only if it satisfies (in a certain domain of its arguments)

$$\dot{F} \equiv \frac{\partial F}{\partial t} + \sum_{k=0}^{\infty} v_{\alpha A_1 \cdots A_k} \frac{\partial F}{\partial r_{\alpha A_1 \cdots A_k}} + \sum_{k=0}^{\infty} \dot{v}_{\alpha A_1 \cdots A_k} \frac{\partial F}{\partial v_{\alpha A_1 \cdots A_k}}$$
$$+ \sum_{k=0}^{\infty} \dot{\phi}_{i_1 \cdots i_k} \frac{\partial F}{\partial \phi_{i_1 \cdots i_k}} = 0 \tag{2.7}$$

in which we have to substitute from (2.6) for the corresponding quantities.

In the remainder of this paper we shall find all the solutions of equation

(2.7). It is a single equation. But with a function F of a finite number of arguments the coefficients in (2.7) contain some of the basic quantities which are not among the arguments of F(!). Since these are arbitrary in their domains it follows that (2.7) in fact decomposes into a system of equations.

3. *The DCMs in Newton's Theory with no Restrictions*

We first state and explain the final result of this section; then we shall prove it.

Let us define the differential quantities

$$K^{(N)}_{[AB]} = K^{(N)}_{AB} \equiv (\mathbf{r}_A \cdot \boldsymbol{v}_B) - (\mathbf{r}_B \cdot \boldsymbol{v}_A) \tag{3.1}$$

[The superscript (N) emphasizes that we are dealing with Newton's theory.] Given any $\mathbf{r}(t;d)$, the $K^{(N)}_{AB}$ are functions of t and d. The same holds for the quantities

$$K^{(N)}_{ABC_1 \cdots C_n} \equiv K^{(N)}_{AB/C_1/C_2/\cdots/C_n} \qquad (n = 1, 2, \ldots) \tag{3.2}$$

and, clearly, they are also differential quantities. It is easy to show (by induction) that for a given n the $K^{(N)}_{ABC_1 \cdots C_n}$ are of order $n+1$, and as functions of the basic quantities they depend only on the $\{\mathbf{r}_{A_1 \cdots A_k}, \mathbf{v}_{A_1 \cdots A_k}\}_{k=1}^{n+1}$.

The main result of this section is as follows. *Apart from the symmetries*

$$K^{(N)}_{A_1 \cdots A_n} = K^{N}_{[A_1 A_2](A_3 \cdots A_n)}, \qquad K^{(N)}_{[A_1 A_2 A_3] A_4 \cdots A_n} = 0$$
$$(n = 2, 3, \ldots) \tag{3.3}$$

the $K^{(N)}_{A_1 \cdots A_n}$ *are functionally independent and form a basis for the DCMs in Newton's theory with no restrictions. Also, they are all covariant.* We shall refer to the $K^{(N)}_{A_1 \cdots A_n}$ as the *basic DCMs.*

In the remainder of the paragraph we prove this assertion. The proof consists of four subparagraphs.

3.1. *A DCM is Never a Function of the* $\{r_\alpha, v_\alpha, \phi, \phi_i, \phi_{ij}, \ldots\}$. Assume that the highest derivatives of ϕ occurring among the arguments of a DCM, F, are of order K. We prove immediately that F, then, cannot be at all a nontrivial function of the $\{\phi_{i_1 \ldots i_K}\}$, and thus, by induction, F is not at all a function of ϕ and of its derivatives. To prove that F is independent of the $\{\phi_{i_1 \cdots i_K}\}$ we perform another inductive process. At first we should note that, although F is not a function of the $\{\phi_{i_1 \cdots i_{K+1}}\}$ (by assumption) these quantities do appear in (2.7) (in the expressions for $\{\dot{v}_{\alpha A_1 \cdots A_l}\}_{l \geqslant K}$ and $\dot{\phi}_{i_1 \cdots i_K}$). However, the $\phi_{i_1 \cdots i_{K+1}}$ with one index at least zero, the $\phi_{i_1 \cdots i_K 0}$, appear only in the $\dot{\phi}_{i_1 \cdots i_K}$, that is, in the terms

$$(\phi_{i_1 \cdots i_K 0} + \phi_{i_1 \cdots i_K \alpha} v_\alpha) \frac{\partial F}{\partial \phi_{i_1 \cdots i_K}}$$

Let us assume by induction that F is independent of the $\phi_{i_1 \cdots i_K}$ with at least $r+1$ ($r < K$) indices zero, (the $\phi_{0 \cdots 0 i_{r+2} \cdots i_K}$). This is equivalent to the equations $\psi_{i_1 \cdots i_K}(\partial F/\partial \phi_{i_1 \cdots i_K}) = 0$ for all the constants $\psi_{i_1 \cdots i_K} =$

$\psi_{(i_1 \ldots i_K)}$, the nonvanishing components of which possess at least $r+1$ indices zero. Now, let us define $\psi^{(r)}_{i_1 \ldots i_K} \equiv \phi_{i_1 \ldots i_K 0}$ if exactly r indices among the $i_1 \ldots, i_K$ vanish and otherwise $\psi^{(r)}_{i_1 \ldots i_K} \equiv 0$. Then, it is easy to show that the $\phi_{i_1 \ldots i_{K+1}}$ with exactly $r+1$ indices zero appear in (2.7) only in the expression $\psi^{(r)}_{i_1 \ldots i_K}(\partial F/\partial \phi_{i_1 \ldots i_K})$. This expression should vanish since (2.7) should hold and the $\psi^{(r)}_{i_1 \ldots i_K}$ obtained, while changing the $\phi_{i_1 \ldots i_{K+1}}$, are arbitrary apart from $\psi^{(r)}_{i_1 \ldots i_K} = \psi^{(r)}_{(i_1 \ldots i_K)}$ and $\psi^{(r)}_{i_1 \ldots i_K} = 0$ if not exactly r of the $i_1, \ldots, i_K$ are zero, and this, in turn, means that F is also independent of the $\phi_{i_1 \ldots i_K}$ with exactly r indices zero. Therefore F is independent of the $\phi_{i_1 \ldots i_K}$ with at least r indices zero and the inductive process can be continued. This completes the proof that F is independent of ϕ and of its derivatives. Nevertheless, derivatives of ϕ still appear in (2.7). In particular ϕ_α appear only in the terms $\phi_\alpha(\partial F/\partial v_\alpha)$ and their arbitrariness and (2.7) imply $(\partial F/\partial v_\alpha) = 0$. Then the v_α occur in (2.7) only in the term $v_\alpha(\partial F/\partial r_\alpha)$ and by the same procedure $(\partial F/\partial r_\alpha) = 0$, too.

Therefore, among the basic quantities, the $\{r_\alpha, v_\alpha, \phi, \phi_i, \phi_{ij}, \ldots\}$ are not available for the construction of DCMs. We shall call

$$\{t, r_{\alpha A}, v_{\alpha A}, r_{\alpha AB}, v_{\alpha AB}, \ldots\} \tag{3.4}$$

the available basis, and refer to its members as the *available basic quantities*, and to functions of these arguments as the *available differential quantities.*

3.2 *Modifications of the available basis (3.4)*. We shall change the available basis (3.4) in a way that leaves the available differential quantities unchanged. Given $\mathbf{r}_A, \mathbf{v}_A$ which satisfy (2.5), the $\{\langle(r_A), -(v_A)\rangle\}_{A=\bar{1}}^{\bar{6}}$ form a linear basis in the space of 6-tuples. Let us consider the nth-order members of the available basis, $v_{\alpha A_1 \ldots A_n}, r_{\alpha A_1 \ldots A_n}$. For given $A_1, \ldots, A_n$, $\langle \mathbf{v}_{A_1 \ldots A_n}, \mathbf{r}_{A_1 \ldots A_n}\rangle$ is an *arbitrary* 6-tuple [remember (2.3)]. Therefore, it is fixed by its six 6-Cartesian scalar products with the members of the above-mentioned basis,

$$h_{AA_1 \ldots A_n} \equiv (\mathbf{r}_A \cdot \mathbf{v}_{A_1 \ldots A_n}) - (\mathbf{v}_A \cdot \mathbf{r}_{A_1 \ldots A_n}) \qquad (A = \bar{1}, \ldots, \bar{6}) \tag{3.5}$$

which are also arbitrary (for given $A_1, \ldots, A_6$). We may modify the basis (3.4) by replacing $\mathbf{v}_{A_1 \ldots A_n}, \mathbf{r}_{A_1 \ldots A_n}$ by $h_{AA_1 \ldots A_n}$. The modified basis is

$$\{t, r_{\alpha A}, v_{\alpha A}, h_{ABC}, h_{ABCD}, \ldots\} \tag{3.6}$$

The quantities appearing in this basis are functionally independent apart from the symmetries

$$h_{A_1 A_2 \ldots A_n} = h_{A_1 (A_2 \ldots A_n)} \qquad (n = 3, 4, \ldots) \tag{3.7}$$

Also it is worth noting that the quantities $\{t, r_{\alpha A}, v_{\alpha A}\} U \{h_{A_1 \ldots A_n}\}_{n=3}^{N+1}$, form a basis for the Nth-order available differential quantities ($N = 2, 3, \ldots$). In the forthcoming further modifications of the basis (3.6) we shall try to preserve an analogous property.

We now apply Lemma 1 of the Appendix to every $h_{A_1 \cdots A_n}$ in (3.6), replacing it by the pair $\langle h_{[A_1A_2]A_3 \cdots A_n}, h_{(A_1 \cdots A_n)}\rangle$. The pair components are arbitrary apart from the obvious symmetries and

$$h_{[A_1A_2]A_3 \cdots A_n} = h_{[A_1A_2](A_3 \cdots A_n)} \qquad h_{[[A_1A_2]A_3]A_4 \cdots A_n} = 0$$
$$(n = 3, 4, 5, \ldots) \tag{3.8}$$

By an inductive process we may add to each quantity in the basis a term with the same symmetry properties which is of a lower order. This we shall do presently.

We first replace every $h_{(A_1 \cdots A_n)}$, $(n = 3, 4, 5, \ldots)$, by

$$H_{A_1 \cdots A_n} \equiv h_{(A_1A_2A_3/A_4/ \cdots /A_n)} \qquad (n = 3, 4, \ldots) \tag{3.9}$$

Indeed, $H_{A_1 \cdots A_n}$ has the symmetries of $h_{(A_1 \cdots A_n)}$,

$$H_{A_1 \cdots A_n} = H_{(A_1 \cdots A_n)} \tag{3.10}$$

and they differ from each other by a quantity of order n-2 at most, as follows by induction from (3.9) and (3.5).

Secondly, we replace every $h_{[A_1A_2]A_3 \cdots A_n}$, $(n = 3, 4, \ldots)$, by $K^{(N)}_{A_1 \cdots A_n}$. Indeed $K^{(N)}_{A_1 \cdots A_n}$ has the symmetries (3.8) of $h_{[A_1A_2]A_3 \cdots A_n}$ [equation (3.3)]. These symmetries are implied by (3.1), (3.2), and by $K^{(N)}_{[A_1A_2A_3]} = 0$. This last symmetry follows from the explicit expression

$$K^{(N)}_{A_1A_2A_3} = (\mathbf{r}_{A_1} \cdot \mathbf{v}_{A_2A_3}) - (\mathbf{r}_{A_2} \cdot \mathbf{v}_{A_1A_3}) + (\mathbf{r}_{A_1A_3} \cdot \mathbf{v}_{A_2}) - (\mathbf{r}_{A_2A_3} \cdot \mathbf{v}_{A_1})$$

implied also by (3.1) and (3.2). Again equations (3.1) and (3.2) imply (by induction) that $2h_{[A_1A_2]A_3 \cdots A_n}$ is equal to the terms of the highest order, $(n - 1)$, in the explicit expression $K^{(N)}_{A_1 \cdots A_n}$, so that $2h_{[A_1A_2]A_3 \cdots A_n}$ and $K^{(N)}_{A_1 \cdots A_n}$ differ from each other by terms of lower order, as should be. An important result of this is [with (3.8) in mind] that apart from (3.3) the $K^{(N)}_{A_1 \cdots A_n}$ are arbitrary.

Thus, the last form of the available basis we shall adopt is

$$\{t, \mathbf{r}_{\alpha A}, v_{\alpha A}, H_{ABC}, K^{(N)}_{ABC}, H_{ABCD}, K^{(N)}_{ABCD}, \ldots\} \tag{3.11}$$

its members are functionally independent apart from the symmetries (3.3) and (3.10). Also the quantities $\{t, r_{\alpha A}, v_{\alpha A}\} U \{H_{A_1 \cdots A_k}; K^{(N)}_{A_1 \cdots A_k}\}_{k=3}^{n+1}$ form a basis for the nth-order available differential quantities $(n = 2, 3, \ldots)$.

3.3. *DCMs of the First Order* $[F(t, r_{\alpha A}, v_{\alpha A})]$. Equation (2.7) for such a function takes the form

$$\frac{\partial F}{\partial t} + v_{\alpha A}\left(\frac{\partial F}{\partial r_{\alpha A}}\right) + \phi_{\alpha\beta} r_{\alpha A}\left(\frac{\partial F}{\partial v_{\beta A}}\right) = 0 \tag{3.12}$$

Since the symmetric $\phi_{\alpha\beta}$ are arbitrary, (3.12) is equivalent to the system

$$\text{(a)} \qquad \frac{\partial F}{\partial t} + v_{\alpha A}\left(\frac{\partial F}{\partial r_{\alpha A}}\right) = 0$$

$$\text{(b)} \qquad r_{\alpha A}\left(\frac{\partial F}{\partial v_{\beta A}}\right) + r_{\beta A}\left(\frac{\partial F}{\partial v_{\alpha A}}\right) = 0$$

Now we apply to these equations the crossing process as follows.
[a, b] = (c), [a, c] = (d)

$$\text{(d)} \qquad v_{\alpha A}\left(\frac{\partial F}{\partial r_{\beta A}}\right) + v_{\beta A}\left(\frac{\partial F}{\partial r_{\alpha A}}\right) = 0$$

Applying (d) to (a) leads to

$$\text{(e)} \qquad \frac{\partial F}{\partial t} = 0$$

Contraction of an appropriate pair of indices in [b, d] leads to

$$\text{(f)} \qquad r_{\alpha A}\left(\frac{\partial F}{\partial r_{\beta A}}\right) - v_{\beta A}\left(\frac{\partial F}{\partial v_{\alpha A}}\right) = 0$$

Therefore, the initial system {(a), (b)} implies and is implied by the system

$$\text{(b)} \qquad r_{\alpha A}\left(\frac{\partial F}{\partial v_{\beta A}}\right) + r_{\beta A}\left(\frac{\partial F}{\partial v_{\alpha A}}\right) = 0$$

$$\text{(d)} \qquad v_{\alpha A}\left(\frac{\partial F}{\partial r_{\beta A}}\right) + v_{\beta A}\left(\frac{\partial F}{\partial r_{\alpha A}}\right) = 0$$

$$\text{(f)} \qquad r_{\alpha A}\left(\frac{\partial F}{\partial r_{\beta A}}\right) - v_{\beta A}\left(\frac{\partial F}{\partial v_{\alpha A}}\right) = 0$$

$$\text{(e)} \qquad \frac{\partial F}{\partial t} = 0$$

It is easy to show that this system is closed, (That is, the crossing process does not lead to new linearly independent equations). F is not a function of t [because of (e)], and it can be any function of the 36 variables $\{r_{\alpha A}, v_{\alpha A}\}$ that satisfies (b), (d), and (f). This last system consists of at most 21 (= 6+6+9) linearly independent equations. Indeed there are exactly 21 linearly independent equations since the addition of the extra 15 (= 36 − 21) equations

$$r_{\alpha A}\left(\frac{\partial F}{\partial v_{\beta A}}\right) - r_{\beta A}\left(\frac{\partial F}{\partial v_{\alpha A}}\right) = 0 \qquad \text{(3 equations)}$$

$$v_{\alpha A}\left(\frac{\partial F}{\partial r_{\beta A}}\right) - v_{\beta A}\left(\frac{\partial F}{\partial r_{\alpha A}}\right) = 0 \qquad \text{(3 equations)}$$

$$r_{\alpha A}\left(\frac{\partial F}{\partial r_{\beta A}}\right) + v_{\beta A}\left(\frac{\partial F}{\partial v_{\alpha A}}\right) = 0 \qquad \text{(9 equations)}$$

to them imples $(\partial F/\partial r_{\alpha A}) = (\partial F/\partial v_{\alpha A}) = 0$. This fact is a direct result of all these equations since (2.5) holds. [We use (2.5) in the form that $r_{\alpha A}\psi_A = v_{\alpha A}\psi_A = 0$ imply $\psi_A = 0$.] Therefore, there exist exactly 15 (= 36 − 21), functionally independent DCMs of the first order, which serve as the basis for the DCMs of the first order. We may choose for them the 15 $\{K_{AB}^{(N)}\}$ defined by (3.1). Indeed these are DCMs as implied by (3.1) and (2.6). These are also 15 functionally independent functions, since it is easy to show with the aid of (2.5) that the only solution, $X_{AB} = X_{[AB]}$, of the equations $X_{AB}(\partial K_{AB}^{(N)}/\partial r_{\alpha C}) = X_{AB}(\partial K_{AB}^{(N)}/\partial v_{\alpha C}) = 0$ ($\alpha = 1, 2, 3$; $C = \overline{1}, \ldots, \overline{6}$) is the trivial one.

3.4. *DCMs of High Orders.* The basic available differential quantities are given by (3.11). We have to know their time derivatives along the particles' histories. Those of t, $r_{\alpha A}$, $v_{\alpha A}$ are given by (2.6). Since the $\{K_{AB}^{(N)}\}$ are DCMs it follows from the definition (3.2) that the $\{K_{A_1}^{(N)}\ldots_{A_n}\}$, too, are DCMs:

$$\dot{K}_{A_1}^{(N)}\ldots_{A_n} = 0 \qquad (n = 2, 3, \ldots) \tag{3.13}$$

Now, only the $\{\dot{H}_{A_1}\ldots_{A_n}\}$ are as yet unknown. Equations (3.5) and (2.6) lead to

$$\dot{h}_{A_1A_2A_3} = \phi_{\alpha\beta\gamma} r_{\alpha A_1} r_{\beta A_2} r_{\gamma A_3} \tag{3.14}$$

Therefore,.the definition (3.9) implies

$$\dot{H}_{A_1}\ldots_{A_n} = \sum_{k=3}^{n-1} \phi_{\alpha_1}\ldots_{\alpha_k}\psi_{\alpha_1}\ldots_{\alpha_k A_1}\ldots_{A_n} + \phi_{\alpha_1}\ldots_{\alpha_n} r_{\alpha_1 A_1}\ldots r_{\alpha_n A_n} \tag{3.15}$$

where the $\psi\ldots$,'s are certain functions of the $\{r_{\alpha A}, r_{\alpha AB}, r_{\alpha ABC}, \ldots\}$.

Let F be a DCM. Assume that as a function of the basic quantities (3.11), the $\{H_{A_1}\ldots_{A_n}\}$ for some n ($n \geqslant 3$) occur among its arguments while the $\{H_{A_1}\ldots_{A_k}\}_{k>n}$ do not. F should satisfy $\dot{F} = 0$. From equations (3.13), (3.15), and (2.6) it follows that this equation has the form of a linear inhomogeneous polynomial in the $\{\phi_{\alpha_1}\ldots_{\alpha_l}\}$. Since, apart from symmetry, the $\{\phi_{\alpha_1}\ldots_{\alpha_l}\}$

are arbitrary (and F is not a function of them) it follows that all the monomials of this polynomial vanish identically. In particular, with the aid of (3.15),

$$\frac{\partial F}{\partial t} + v_{\alpha A}\left(\frac{\partial F}{\partial r_{\alpha A}}\right) = 0 \tag{3.16a}$$

$$\phi_{\alpha\beta} r_{\alpha A}\left(\frac{\partial F}{\partial v_{\beta A}}\right) = 0 \tag{3.16b}$$

$$\phi_{\alpha_1 \cdots \alpha_n} r_{\alpha_1 A_1} \cdots r_{\alpha_n A_n}\left(\frac{\partial F}{\partial H_{A_1 \cdots A_n}}\right) = 0 \tag{3.16c}$$

where the $\phi_{\alpha_1 \cdots \alpha_k}$ are considered as arbitrary symmetric constants. (For every set of symmetric constants we get an equation.) [F should satisfy even more equations than these (!).]

It would be convenient to separate the indices of type $\{\alpha, \beta, \ldots\}$ from those of type $\{A, B, \ldots\}$, in order that it be possible to note symmetrization, for example, of one type of indices by the appropriate bracket notation. In the remainder of this subsection we shall raise all the α-type indices and write them as upper indices, without any change of the quantities. (For example, $r_A^\alpha \equiv r_{\alpha A}$). With this notation equations (3.16) take the form

(a) $$\frac{\partial F}{\partial t} + v_A{}^\alpha\left(\frac{\partial F}{\partial r_A{}^\alpha}\right) = 0$$

(b) $$r_A{}^\alpha\left(\frac{\partial F}{\partial v_A{}^\beta}\right) + r_A{}^\beta\left(\frac{\partial F}{\partial v_A{}^\alpha}\right) = 0$$

(c) $$r_{(A}^{\alpha_1} \cdots r_{An)}^{\alpha n}\left(\frac{\partial F}{\partial H_{A_1 \cdots An}}\right) = 0$$

We note that equations (a) and (b) are identical to (a) and (b) of Section 3.3 and imply therefore

(d) $$v_A{}^\alpha\left(\frac{\partial F}{\partial r_A{}^\beta}\right) + v_A{}^\beta\left(\frac{\partial F}{\partial r_A{}^\beta}\right) = 0$$

[which is the same as (d) of 3.3].

We prove now by an inductive process that for every k ($k = 0, 1, \ldots, n$)

(e) $$v_{(A_1}^{\alpha_1} \cdots v_{Ak}^{\alpha k} r_{Ak+1}^{\alpha k+1} \cdots r_{An)}^{\alpha n}\left(\frac{\partial F}{\partial H_{A_1 \cdots An}}\right) = 0$$

For $k = 0$ (e) is identical with (c) and is therefore correct. Assume that (e) is correct for a certain k ($k = 0, 1, \ldots, n-1$); then we perform [d, e] = (f);

contraction of $\alpha = \alpha_{k+1}$ at (f) leads to (g); again, contraction of $\beta = \alpha_{k+2}$ at (g) leads to (h),

$$\text{(h)}\quad v^{\alpha}_{(A_1} \cdots v^{\alpha k}_{Ak} v^{\eta}_{Ak+1} r^{\eta}_{Ak+2} r^{\alpha k+3}_{Ak+3} \cdots r^{\alpha n}_{An)} \left(\frac{\partial F}{\partial H_{A_1} \cdots {}_{An}}\right) = 0$$

and substitution from (h) to (g) leads to (e) for $k + 1$. Therefore (e) is correct for all $k = 0, 1, \ldots, n$. We may write (e) in the form

$$\text{(e*)}\quad v^{\alpha_1}_{A_1} \cdots v^{\alpha k}_{Ak} r^{\alpha k+1}_{Ak+1} \cdots r^{\alpha n}_{An} \left(\frac{\partial F}{\partial H_{(A_1} \cdots {}_{An)}}\right) = 0 \qquad (k = 0, \ldots, n)$$

where in $(\partial F/\partial H_{(A_1} \cdots {}_{An)})$ differentiation is *followed* by symmetrization (there is in fact no other meaning). It is obvious that a changing of the order of the $\{r_A{}^\alpha, v_A{}^\alpha\}$ appearing in (e*) always leads to a correct equation. Therefore with the aid of (2.5) and the fact that (e*) is correct for every k ($k = 0, \ldots, n$), we obtain $(\partial F/\partial H_{(A_1} \cdots {}_{An)}) = 0$, or $S_{A_1} \cdots {}_{An} (\partial F/\partial H_{A_1} \cdots {}_{An}) = 0$ for all symmetric constants $S_{A_1} \cdots {}_{An}$. Since the $H_{A_1} \cdots {}_{An}$ are symmetric, too, this means that F cannot be a nontrivial function of the $H_{A_1} \cdots {}_{An}$. By induction this is correct for every n, ($n = 3, 4, 5, \ldots$). Now we are left with equations (a) and (b) only, which are identical with (a) and (b) of Section 3.3. and imply that F may depend on the $\{t, r_A{}^\alpha, v_A{}^\alpha\}$ only through the $K^{(N)}_{AB}$. This completes the proof of the assertion at the beginning of this section.

4. *The DCMs in Newton's Theory in Vacuum*

Every DCM of Newton's theory with no restrictions is obviously a DCM in vacuum. Therefore, the DCMs set in vacuum may be larger than the set of DCMs that are good for all the possible gravitational fields. On the other hand, the vacuum condition reduces the set of differential quantities, since it introduces, apart from (2.3), further restrictions on the basis of differential quantities (2.4). These are

$$\phi_{i_1} \cdots {}_{ik\alpha\alpha} = 0 \qquad (k = 0, 1, 2, \ldots) \tag{4.1}$$

Therefore, at least in principle, the restrictions (4.1) may make some of the DCMs of Newton's theory with no restrictions trivial. However, what really happens is that *the vacuum condition does not change the set of DCMs at all.* We outline the proof right now.

Our aim is to show that the $\{K^{(N)}_{A_1} \cdots {}_{An}\}^{\infty}_{n=2}$, again, form a basis for the DCMs in vacuum. We shall follow the proof of Section 3, but from time to time we shall have to overcome the new difficulties caused by the further restrictions (4.1). In particular we adopt Section 3.1 almost word for word and derive the same result, that is, a DCM cannot be a nontrivial function of the $\{r_\alpha, v_\alpha, \phi, \phi_i, \phi_{ij} \ldots\}$, and, thus, the available basis is again (3.3). Then we apply to it the modifications of 3.2 and arrive again at the final modified available basis (3.11). Two problems, however, should be discussed in detail.

4.1. *DCMs of the First Order* $[F(t, r_{\alpha A}, v_{\alpha A})]$ (*Vacuum Case*). Equation (3.12) now leads to the following system only:

$$\text{(a)} \qquad \frac{\partial F}{\partial t} + v_{\alpha A}\left(\frac{\partial F}{\partial r_{\alpha A}}\right) = 0$$

$$\text{(b*)} \qquad \phi_{\alpha\beta} r_{\alpha A}\left(\frac{\partial F}{\partial v_{\beta A}}\right) = 0$$

for every symmetric set of constants $\phi_{\alpha\beta}$ which satisfy the further condition $\phi_{\alpha\alpha} = 0$. Equation (b*) is equivalent to

$$r_{\alpha A}\left(\frac{\partial F}{\partial v_{\beta A}}\right) + r_{\beta A}\left(\frac{\partial F}{\partial v_{\alpha A}}\right) + \lambda\delta_{\alpha\beta} = 0$$

for some λ. Contraction of $\alpha = \beta$ implies $\lambda = -\frac{2}{3} r_{\gamma C}(\partial F/\partial v_{\gamma C})$. Hence, (b*) is equivalent to

$$\text{(b)} \qquad r_{\alpha A}\left(\frac{\partial F}{\partial v_{\beta A}}\right) + r_{\beta A}\left(\frac{\partial F}{\partial v_{\alpha A}}\right) - \frac{2}{3}\delta_{\alpha\beta} r_{\gamma C}\left(\frac{\partial F}{\partial v_{\gamma C}}\right) = 0$$

Now we perform the following operations: [a, b] = (c); replacing of the indices (α, β) in (c) by (γ, δ) and then [b, c] = (d); with the aid of (b) it turns out that (d) is equivalent to

$$\text{(d*)} \qquad \left(\frac{4}{3}\delta_{\gamma\alpha}\delta_{\delta\beta} + \frac{4}{3}\delta_{\alpha\delta}\delta_{\gamma\beta} - \frac{8}{9}\delta_{\alpha\beta}\delta_{\gamma\delta}\right) r_{\mu A}\left(\frac{\partial F}{\partial v_{\mu A}}\right) = 0$$

contraction of $\alpha = \gamma$ and $\delta = \beta$ in (d*) leads to $r_{\mu A}(\partial F/\partial v_{\mu A}) = 0$; substitution of this equation in (b) finally implies

$$\text{(e)} \qquad r_{\alpha A}\left(\frac{\partial F}{\partial v_{\beta A}}\right) + r_{\beta A}\left(\frac{\partial F}{\partial v_{\alpha A}}\right) = 0$$

Now, (a) and (e) are identical with (a) and (b) of Section 3.3 and, as was pointed out there, they lead to the result that F may depend on the $\{t, r_{\alpha A}, v_{\alpha A}\}$ only through the $K_{AB}^{(N)}$.

4.2. *DCMs of High Orders* (*Vacuum Case*). As in Section 3.4 we obtain the result that if F is a DCM such that, as a function of the basic quantities (3.11), the $\{H_{A_1 \cdots A_n}\}$ $(n \geqslant 3)$ occur among its arguments while the $\{H_{A_1 \cdots A_k}\}_{k>n}$ do not, then F satisfies equations (3.16) (among others); but now the constants $\{\phi_{\alpha_1 \cdots \alpha_k}\}$ appearing there are more restricted – by (4.1) also.

Equations (3.16a) and (3.16b) are identical to (a), (b*) of Section 4.1, and, as was shown there, they imply (and are implied by)

$$\text{(a)} \qquad \frac{\partial F}{\partial t} + v_A{}^{\alpha}\left(\frac{\partial F}{\partial r_A{}^{\alpha}}\right) = 0$$

(b) $$r_A{}^\nu\left(\frac{\partial F}{\partial v_A{}^\mu}\right)+r_A{}^\mu\left(\frac{\partial F}{\partial v_A{}^\nu}\right)=0$$

Equation (3.16c) takes the form

(c) $$\phi^{\alpha_1\cdots\alpha_n}r_{A_1}^{\alpha_1}\cdots r_{An}^{\alpha n}\left(\frac{\partial F}{\partial H_{A_1\cdots An}}\right)=0$$

where the $\{\phi^{\alpha_1\cdots\alpha_n}\}$ are arbitrary symmetric constants that have to satisfy

(d) $$\phi^{\mu\mu\alpha_3\cdots\alpha_n}=0$$

In equations (a), (b), (c), (d) we have used the convention of Section 3.4, that is, raising of all the α-type indices.

We apply to equations (a), (b), (c), dismissing for the moment equation (d), the following process: [a, c] = (e), [b, e] = (f)

(f) $$\delta^{\alpha_1(\mu}\phi^{\nu)\alpha_2\cdots\alpha_n}r_{(A_1}^{\alpha_1}\cdots r_{An)}^{\alpha n}\left(\frac{\partial F}{\partial H_{A_1\cdots An}}\right)=0$$

$$(\mu,\nu=1,2,3)$$

and these equations are equivalent to

(g*) $$S^{\mu\nu}\delta^{\alpha_1\mu}\phi^{\nu\alpha_2\cdots\alpha_n}r_{(A_1}^{\alpha_1}\cdots r_{An)}^{\alpha n}\left(\frac{\partial F}{\partial H_{A_1\cdots An}}\right)=0$$

for arbitrary symmetric $S^{\mu\nu}$. Therefore, if F satisfies equations (a), (b), and (c) with the symmetric constants $\phi^{\alpha_1\cdots\alpha_n}$, it also satisfies

(g) $$S^{\mu(\alpha_1}\phi^{\alpha_2\cdots\alpha_n)\mu}r_{A_1}^{\alpha_1}\cdots r_{An}^{\alpha n}\left(\frac{\partial F}{\partial H_{A_1\cdots An}}\right)=0$$

for arbitrary symmetric constants $S^{\mu\nu}$. [In fact, (g) and (g*) are identical.]

This discussion proves the following result: If F satisfies (c) for the symmetric $\{\phi^{\alpha_1\cdots\alpha_n}\}$ in a certain linear space, then F also satisfies (c) for the symmetric $\{\phi^{\alpha_1\cdots\alpha_n}\}$ in a larger linear space, the space spanned linearly by the former one and by the set $\{S^{\mu(\alpha_1}\phi^{\alpha_2\cdots\alpha_n)\mu}\}$, where the $S^{\mu\nu}$ are arbitrary symmetric and the $\{\phi^{\alpha_1\cdots\alpha_n}\}$ are, again, in the former space.

We now apply this result and Lemma 2 of the appendix to equations (c) and (d). It follows that since F satisfies (c) with the $\phi^{\alpha_1\cdots\alpha_n}$, which vanish by one contraction of indices [remember (d)], it satisfies (c) also with the $\{\phi^{\alpha_1\cdots\alpha_n}\}$, which vanish by two contractions, and therefore, again, F satisfies (c) with the $\{\phi^{\alpha_1\cdots\alpha_n}\}$, which vanish by three contractions, and so on. Thus, finally, F satisfies (c) with the symmetric $\phi^{\alpha_1\cdots\alpha_n}$ with no more restrictions. The situation now is completely the same as in Section 3.4, and the path to the desired assertion of this section is clear.

5. *Some Concluding Remarks*

We emphasize again that from the point of view of the DCMs there is no difference between the general theory of Newton and the vacuum case of this theory. All the DCMs are the $\{K^{(N)}_{A_1 \cdots A_n}\}_{n=2}^{\infty}$ defined by (3.1) and (3.2).

Of course, all the DCMs are covariant (Galilei invariant, particularly). Hence, there is no way of determining the displacement, velocity, and orientation relative to a fixed Galilean frame, of a given laboratory in free fall from (local) measurements of differential quantities; a fact worthy of attention.

It is possible to show that the constancy of the $K^{(N)}_{AB}$ defined by (3.1) along the particles' motions is equivalent to the existence of the first integral invariants of Poincare, known from classical mechanics (Goldstein, 1962), and it constitutes a generalization of the circulation theorem known from hydrodynamics (Landau and Lipshitz, 1959). The existence of any other DCM is implied by the $K^{(N)}_{AB}$ according to Section 3.4; its values, however, may be independent. The main work was to demonstrate that no other DCMs exist. (The other integral invariants of Poincaré have also a differential formulation, which is always a set of DCMs of the first order. These should be functions of the $K^{(N)}_{AB}$.)

A treatment of the analogous problem in the frame of Einstein's theory appears in the paper following this one (Enosh and Kovetz, 1978).

Appendix

Lemma 1. Given any k ($k = 3, 4, 5, \ldots$), there exists a linear isomorphism between the linear space

$$\{R_{A_1 \cdots A_k} : R_{A_1 \cdots A_k} = R_{A_1 (A_2 \cdots A_k)}\}$$

and the linear space of the pairs

$$\{\langle T_{A_1 \cdots A_k}, S_{A_1 \cdots A_k}\rangle : T_{A_1 \cdots A_k} = T_{[A_1 A_2](A_3 \cdots A_k)},$$
$$T_{[A_1 A_2 A_3] A_4 \cdots A_k} = 0, S_{A_1 \cdots A_k} = S_{(A_1 \cdots A_k)}\}$$

This isomorphism is given by

$$T_{A_1 \cdots A_k} = R_{[A_1 A_2] A_3 \cdots A_k}, \qquad S_{A_1 \cdots A_k} = R_{(A_1 \cdots A_k)}$$

$$R_{A_1 \cdots A_k} = S_{A_1 \cdots A_k} + \frac{2(k-1)}{k} T_{A_1 (A_2 \cdots A_k)}$$

(Here, the A-type indices may run over any finite set.)

We omit the proof.

In order to formulate and prove the next lemma it is convenient to introduce the following definition.

Definition. For every (n, r) ($n = 0, 1, 2, \ldots; r = 1, 2, \ldots, [\frac{1}{2}n] + 1$), $V(n, r)$ is the linear space of the symmetric quantities $\phi_{\alpha_1 \cdots \alpha_n}$ which vanish by r contractions of indices (if $2r > n$, we understand that every $\phi_{\alpha_1} \cdots {}_{\alpha_n}$ vanishes by r contractions). To be more exact,

$$V(n,r) \equiv \begin{cases} \{\phi_{\alpha_1} \cdots {}_{\alpha_n} : \phi_{\alpha_1} \cdots {}_{\alpha_n} = \phi_{(\alpha_1} \cdots {}_{\alpha_n)}, \\ \qquad\qquad \phi_{\mu_1 \mu_1} \cdots {}_{\mu_r \mu_r \alpha_{2r+1}} \cdots {}_{\alpha_n} = 0\}, & r \leqslant [\frac{1}{2}n] \\ \{\phi_{\alpha_1} \cdots {}_{\alpha_n} : \phi_{\alpha_1} \cdots {}_{\alpha_n} = \phi_{(\alpha_1} \cdots {}_{\alpha_n)}\}, & r = [\frac{1}{2}n] + 1 \end{cases}$$

The α-type indices may run over any finite set with more than one element; our convention is $\alpha = 1, \ldots, N \geqslant 2$. ($[\frac{1}{2}n]$ means the integer part of $\frac{1}{2}n$). Thus, for example, $V(0,1) = \{\phi\}$, the space of "scalars"; $V(1,1) = \{\phi_\alpha\}$; $V(2,1) = \{\phi_{\alpha\beta} : \phi_{\alpha\beta} = \phi_{(\alpha\beta)}, \phi_{\alpha\alpha} = 0\}$; $V(2,2) = \{\phi_{\alpha\beta} : \phi_{\alpha\beta} = \phi_{(\alpha\beta)}\}$. Also

$$V(n,1) \subset V(n,2) \subset V(n,3) \subset \cdots \subset V(n, [\tfrac{1}{2}n] + 1)$$

We are now in a position to formulate

Lemma 2. For every (n, r), $(n = 2, 3, 4, \ldots; r = 1, \ldots [\frac{1}{2}n])$, $V(n, r+1)$ is spanned linearly by $V(n, r)$ and by the set $U(n, r)$, where

$$U(n,r) \equiv \{\phi_{\alpha_1} \cdots {}_{\alpha_n} : \phi_{\alpha_1} \cdots {}_{\alpha_n} = S_{\mu(\alpha_1} \psi_{\alpha_2} \cdots {}_{\alpha_n)\mu}, \quad S_{\mu\nu} = S_{(\mu\nu)}, \psi_{\alpha_1} \cdots {}_{\alpha_n} \epsilon V(n,r)\}$$

for $n = 1, 2, \ldots; r = 1, \ldots, [\frac{1}{2}n] + 1$.

In short, $\quad V(n, r+1) = \mathrm{span}\{V(n,r), U(n,r)\} \qquad (n = 2, 3, \ldots; r = 1, \ldots, [\frac{1}{2}n])$

[The restrictions on $S_{\mu\nu}$ may be written as $S_{\mu\nu} \epsilon V(2,2)$.]

Proof. At first we observe that span $U(n,r) = X(n,r)$, where

$$X(n,r) \equiv \{\chi_{\mu(\alpha_1} \cdots {}_{\alpha_n)\mu} : \chi_{\mu\nu\alpha_1} \cdots {}_{\alpha_n} = \chi_{(\mu\nu)(\alpha_1} \cdots {}_{\alpha_n)}, \quad \chi_{\mu_0 \nu_0 \alpha_1} \cdots {}_{\alpha_n} \epsilon V(n,r) \text{ for } \mu_0, \nu_0 = 1, 2, \ldots, N\}$$

for $n = 1, 2, 3, \ldots; r = 1, \ldots, [\frac{1}{2}n] + 1$. This follows from the facts that every $\chi_{\mu\nu\alpha_1} \cdots {}_{\alpha_n}$ with the properties mentioned in the definition of $X(n,r)$ can be represented by a finite linear combination of terms of the type $S_{\mu\nu}\phi_{\alpha_1} \cdots {}_{\alpha_n}$, where $S_{\mu\nu} = S_{(\mu\nu)}$ and $\phi_{\alpha_1} \cdots {}_{\alpha_n} \epsilon V(n,r)$ and that $X(n,r)$ is indeed a linear space. Hence, it is sufficient to prove $V(n, r+1) = \mathrm{span}\{V(n,r), X(n,r)\}$ for $n = 2, 3, 4, \ldots; r = 1, \ldots, [\frac{1}{2}n]$.

One direction, that $V(n, r+1) \supset \mathrm{span}\{V(n,r), X(n,r)\}$, is trivial, since $V(n, r+1) \supset V(n,r)$ and it is obvious from the definitions $V(n, r+1) \supset X(n,r)$, $(n = 1, 2, \ldots; r = 1, \ldots, [\frac{1}{2}n])$.

The opposite direction, that $V(n, r+1) \subset \mathrm{span}\{V(n,r), X(n,r)\}$ is more difficult. This we shall prove by two stages as follows.

Statement 1. For every (n, r) $(n = 2, 3, \ldots; r = 1, 2, \ldots, [\frac{1}{2}n])$, every $\phi_{\alpha_1} \cdots {}_{\alpha_n} \epsilon V(n, r+1)$ is (uniquely) decomposable according to

$$\phi_{\alpha_1} \cdots {}_{\alpha_n} = \delta_{(\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \cdots \delta_{\alpha_{2r-1}\alpha_{2r+1}}\psi_{\alpha_{2r+1}} \cdots {}_{\alpha_n)} + \tau_{\alpha_1} \cdots {}_{\alpha_n}$$

where $\tau_{\alpha_1} \cdots {}_{\alpha_n} \epsilon V(n,r)$ and $\psi_{\alpha_{2r+1}} \cdots {}_{\alpha_n} \epsilon V(n - 2r, 1)$.

Proof. Given $\phi_{\alpha_1 \cdots \alpha_n}$ we perform r contractions of indices on the desired decomposition. Then, the term, the origin of which is $\tau_{\alpha_1 \cdots \alpha_n}$, vanishes and we obtain by induction

$$\phi_{\mu_1\mu_1 \cdots \mu_r\mu_r\alpha_{2r+1} \cdots \alpha_n} = \frac{2^r r!(n-2r-1)!(N+2n-2r-2)!!}{n!(N+2n-4r-2)!!} \psi_{\alpha_{2r+1} \cdots \alpha_n}$$

$[n! \equiv 1 \cdot 2 \cdots n; (2n)!! \equiv 2 \cdot 4 \cdots 2n; (2n+1)!! \equiv 1 \cdot 3 \cdot 5 \cdots (2n+1)]$. Therefore, $\psi \cdots$ and, then also $\tau \cdots$ are determined by $\phi \cdots$ by means of the desired decomposition; moreover, this decomposition exists since $\psi \cdots$ and $\tau \cdots$ thus defined have, indeed, the desired properties. This is a trivial consequence of the definitions.

Statement 2. For every (n, r) $(n = 2, 3, \ldots; r = 1, 2, \ldots, [\frac{1}{2}n]$, and every $\psi_{\alpha_{2r+1} \cdots \alpha_n} \in V(n-2r, 1)$, $\delta_{(\alpha_1\alpha_2} \cdots \delta_{\alpha_{2r-1}\,\alpha_{2r+1} \cdots \alpha_n)} \in X(n, r)$.

Proof. Given any $\psi \cdots \in V(n-2r, 1)$, we have to show how to find $\chi_{\mu\nu\alpha_1 \cdots \alpha_n}$ with the properties

$$\chi_{\mu\nu\alpha_1 \cdots \alpha_n} = \chi_{(\mu\nu)(\alpha_1 \cdots \alpha_n)} \tag{A.1}$$

$$\chi_{\mu\nu\rho_1\rho_1 \cdots \rho_r\rho_r\alpha_{2r+1} \cdots \alpha_n} = 0 \tag{A.2}$$

which is a solution of the equation

$$\chi_{\mu(\alpha_1 \cdots \alpha_n)\mu} = \delta_{(\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \cdots \delta_{\alpha_{2r-1}\,\alpha_{2r}}\psi_{\alpha_{2r+1} \cdots \alpha_n)} \tag{A.3}$$

This we do for three disjoint cases which exhaust the possibilities.

1. $\psi \cdots$ has no indices at all $(n = 2r)$. We guess $\chi \cdots$ of the type

$$\chi_{\mu\nu\alpha_1 \cdots \alpha_{2r}} = (a\delta_{\mu\nu}\delta_{(\alpha_1\alpha_2} \cdots \delta_{\alpha_{2r-1}\,\alpha_{2r})} + b\delta_{(\mu\nu}\delta_{\alpha_1\alpha_2} \cdots \delta_{\alpha_{2r-1}\,\alpha_{2r})})\psi$$

Then (by induction),

$$\chi_{\mu\nu\rho_1\rho_1 \cdots \rho_r\rho_r} = \psi \left[a \frac{(N+2r-2)!!}{(2r-1)!!(N-2)!!} + b \frac{(N+2r)!!}{(2r+1)!!N!!} \right] \delta_{\mu\nu}$$

$$\chi_{\mu(\alpha_1 \cdots \alpha_{2r})\mu} = \psi \left[a + b \frac{N+2r}{2r+1} \right] \delta_{(\alpha_1\alpha_2} \cdots \delta_{\alpha_{2r-1}\alpha_{2r})}$$

Equations (A.1)–(A.3) then imply (for $\psi \neq 0$)

$$a + \frac{N+2r}{(2r+1)N} b = 0, \qquad a + \frac{N+2r}{2r+1} b = 1$$

which have a unique solution,

$$a = -\frac{1}{N-1}, \qquad b = \frac{N(2r+1)}{(N+2r)(N-1)}$$

since $N \geqslant 2$. This means, in particular, that equations (A.1)–(A.3) admit a solution $\chi \cdots$ in case 1.

2. $\psi\cdots$ has only one index ($n = 2r + 1$). We guess $\chi\cdots$ of the type

$$\begin{aligned}\chi_{\mu\nu\alpha_1\cdots\alpha_{2r+1}} = {} & a(\psi_\mu\delta_{(\nu\alpha_{2r+1}}\delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{2r-1}\alpha_{2r})} \\ & + \psi_\nu\delta_{(\mu\alpha_{2r+1}}\delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{2r-1}\alpha_{2r})}) \\ & + b\delta_{\mu\nu}\delta_{(\alpha_1\alpha_2}\cdots\delta_{\alpha_{2r-1}\alpha_{2r}}\psi_{\alpha_{2r+1})} \\ & + c\delta_{(\mu\nu}\delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{2r-1}\alpha_{2r}}\psi_{\alpha_{2r+1})}\end{aligned}$$

Then (by induction)

$$\begin{aligned}\chi_{\mu\nu\rho_1\rho_1\cdots\rho_r\rho_r\alpha} = {} & 2\psi_{(\mu}\delta_{\nu)\alpha}\left[\frac{(N+2r)!!}{(2r+1)!!N!!}a + \frac{(N+2r+2)!!}{(2r+3)!!(N+2)!!}c\right] \\ & + \delta_{\mu\nu}\psi_\alpha\left[\frac{(N+2r)!!}{(2r+1)!!N!!}b + \frac{(N+2r+2)!!}{(2r+3)!!(N+2)!!}c\right]\end{aligned}$$

Equations (A.1)-(A.2) then imply (for $\psi_\alpha \neq 0, N \geqslant 2$) the equations

$$a + \frac{N+2r+2}{(N+2)(2r+3)}c = 0, \qquad b + \frac{N+2r+2}{(N+2)(2r+3)}c = 0$$

Equation (A.3) is equivalent to

$$2a + b + \frac{N+2r-1}{2r}c = 1$$

We obtain by these equations for c

$$\left[\frac{N+2r-1}{2r} - \frac{3(N+2r+2)}{(N+2)(2r+3)}\right]c = 1$$

It is possible to show that the coefficient of c in this equation is strictly positive for $N \geqslant 2$; hence, a solution for a, b, c does exist, which means that equations (A.1)-(A.3) admit a solution $\chi\cdots$ in the second case, too.

3. $\psi\cdots$ has, at least, two indices ($n > 2r + 1$). We guess $\chi\cdots$ of the type

$$\begin{aligned}\chi_{\mu\nu\alpha_1\cdots\alpha_n} = {} & a\delta_{(\alpha_1\alpha_2}\cdots\delta_{\alpha_{2r+1}\alpha_{2r+2}}\psi_{\alpha_{2r+3}\cdots\alpha_n)\mu\nu} \\ & + b(\delta_{(\alpha_1\alpha_2}\cdots\delta_{\alpha_{2r+2}\alpha_{2r+2}}\psi_{\alpha_{2r+3}\cdots\alpha_n\mu)\nu} \\ & + \delta_{(\alpha_1\alpha_2}\cdots\delta_{\alpha_{2r+1}\alpha_{2r+2}}\psi_{\alpha_{2r+3}\cdots\alpha_n\nu)\mu}) \\ & + c\delta_{\mu\nu}\delta_{(\alpha_1\alpha_2}\cdots\delta_{\alpha_{2r-1}\alpha_{2r}}\psi_{\alpha_{2r+1}\cdots\alpha_n)} \\ & + d\delta_{(\mu\nu}\delta_{\alpha_1\alpha_2}\cdots\delta_{\alpha_{2r-1}\alpha_{2r}}\psi_{\alpha_{2r+1}\cdots\alpha_n)}\end{aligned}$$

Then (by induction) $\chi_{\mu\nu\rho_1\rho_1\cdots\rho_r\rho_r\alpha_{2r+1}\cdots\alpha_n} \stackrel{!}{=} \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$, where

$$\mathrm{I} \stackrel{!}{=} a\frac{(N+2n-2r-4)!!\kappa}{n!(N+2n-4r-4)!!}\delta_{(\alpha_{2r+1}\alpha_{2r+2}}\psi_{\alpha_{2r+3}\cdots\alpha_n)\mu\nu}$$

$$\mathrm{II} \stackrel{!}{=} 2b\frac{(N+2n-2r-2)!!\kappa}{(n+1)!(N+2n-4r-2)!!}[(n-2r-1)\delta_{(\alpha_{2r+1}\alpha_{2r+2}}\psi_{\alpha_{2r+3}\cdots\alpha_n)\mu\nu}$$
$$+(\delta_{\mu(\alpha_{2r+1}}\psi_{\alpha_{2r+2}\cdots\alpha_n)\nu}+\delta_{\nu(\alpha_{2r+1}}\psi_{\alpha_{2r+2}\cdots\alpha_n)\mu})]$$

$$\mathrm{III} \stackrel{!}{=} c\frac{(N+2n-2r-2)!!\kappa}{n!(N+2n-4r-2)!!(r+1)}\delta_{\mu\nu}\psi_{\alpha_{2r+1}\cdots\alpha_n}$$

$$\mathrm{IV} \stackrel{!}{=} 2d\frac{(N+2n-2r)!!\kappa}{(n+2)!(N+2n-4r)!!}\Bigg[\delta_{\mu\nu}\psi_{\alpha_{2r+1}\cdots\alpha_n}$$
$$+\frac{(n-2r)(n-2r-1)}{2}\delta_{(\alpha_{2r+1}\alpha_{2r+2}}\psi_{\alpha_{2r+3}\cdots\alpha_n)\mu\nu}$$
$$+(n-2r)(\delta_{\mu(\alpha_{2r+1}}\psi_{\alpha_{2r+2}\cdots\alpha_n)\nu}+\delta_{\nu(\alpha_{2r+1}}\psi_{\alpha_{2r+2}\cdots\alpha_n)\mu})\Bigg]$$

where $\kappa \equiv 2^r(r+1)!(n-2r)!$ It can be shown that equations (A.1)-(A.2) are satisfied by the choice

$$a=\frac{(N+2n-2r)(N+2n-2r-2)(n-2r)(n-2r-1)}{(N+2n-4r-2)(N+2n-4r)(n+1)(n+2)}d$$

$$b=-\frac{(N+2n-2r)(n-2r)}{(N+2n-4r)(n+2)}d, \qquad c=-\frac{2(N+2n-2r)(r+1)}{(N+2n-4r)(n+1)(n+2)}d$$

and, further, that equation (A.3) is satisfied, too, if and only if

$$d\frac{2(r+1)(N+2n-2r)}{(n+1)(n+2)}$$
$$\times\left[\frac{(N+2n-2r-2)(n-2r)(n-2r-1)}{(N+2n-4r-2)(N+2n-4r)n}-\frac{(N+3n-2r)(n-2r)}{(N+2n-4r)n}\right.$$
$$\left.-\frac{1}{N+2n-4r}+1\right]=1$$

and this equation can be solved for d if and only if the expression in the square brackets does not vanish. This is, indeed, the case for $N \geqslant 2$, since this expression is equal to

$$\frac{2r(n-2r)^2+4(N-\frac{3}{2})r(n-2r)+2r(N-1)(N-2)}{(N+2n-4r-2)(N+2n-4r)n}$$

and is strictly positive, because the three terms in the nominator are non-negative while the denominator and the second term in the nominator are strictly positive. Therefore, in case 3, too, equations (A.1)-(A.3) admit a solution $\chi \cdots$. The proof of statement 2 is, thus, completed. This accomplishes the proof of Lemma 2.

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